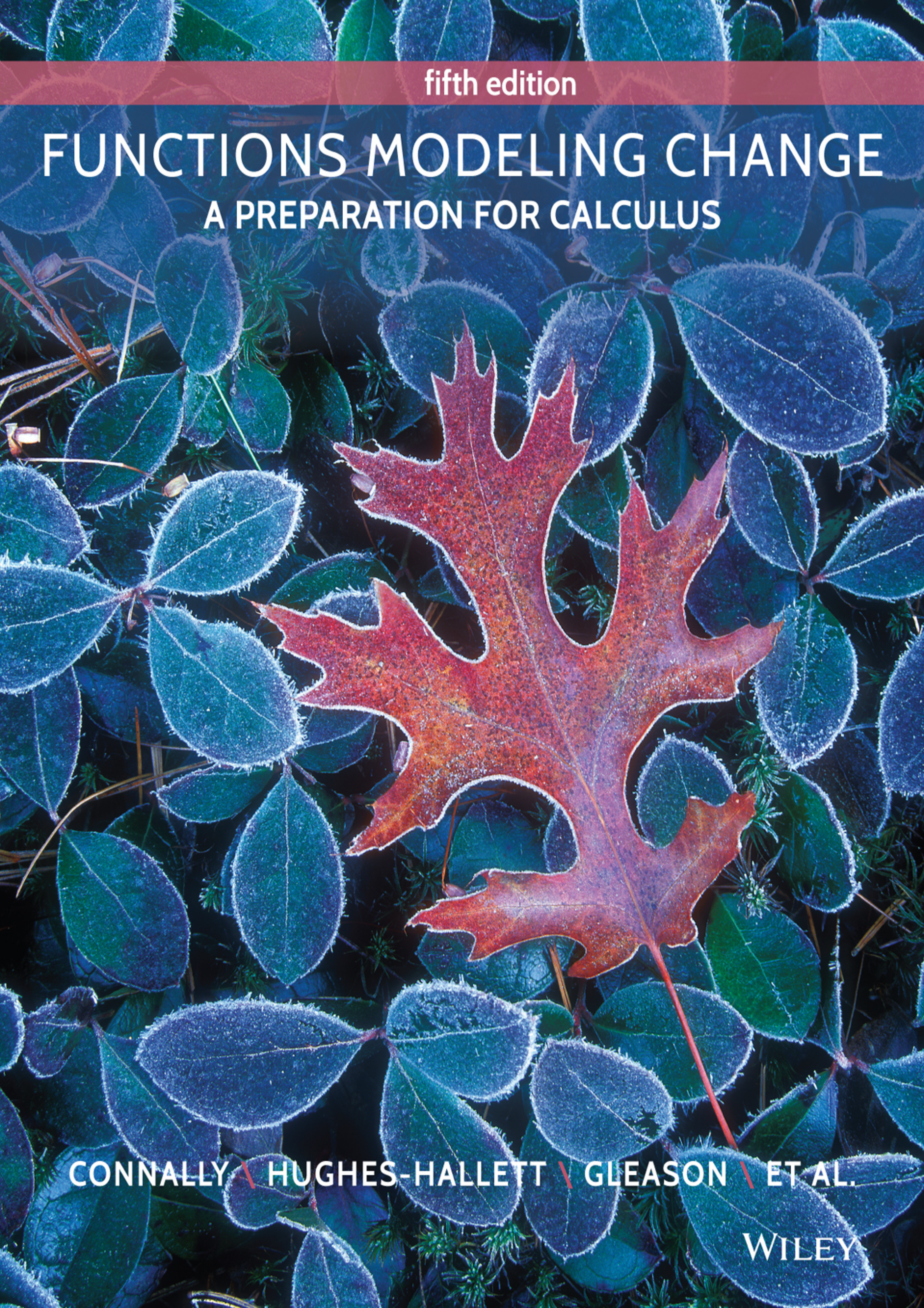
fifth edition

FUNCTIONS MODELING CHANGE

A PREPARATION FOR CALCULUS



CONNALLY \ HUGHES-HALLETT \ GLEASON \ ET AL.

WILEY

FUNCTIONS MODELING CHANGE:
A Preparation for Calculus

Fifth Edition

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Dedicated to Ben, Jonah, and Isabel

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PREFACE

In the 21^{st} century, students need the ability to solve problems; innovation depends on citizens who can think critically. The fifth edition of *Functions Modeling Change: A Preparation for Calculus* reaffirms our effort to refocus the teaching of mathematics in a balanced blend of concepts and procedures. Our approach encourages students to develop their problem-solving skills while acquiring the mathematical background needed to learn calculus or pursue their careers.

Fifth Edition: Focus

This edition of Precalculus continues to stress conceptual understanding and the connections among mathematical ideas. Understanding exponential change, for example, means being able to relate the size of a percent growth to the shape of a function's graph, as well as assessing the reasonability of a model.

Functions are the foundation of much of mathematics. Our approach encourages a robust understanding of functions in traditional and novel contexts. Skills are introduced and reinforced throughout the book. This balance of skills and understanding promotes students' critical thinking.

Fifth Edition: Flexibility

Precalculus courses are taken by a wide range of students and are taught in a wide variety of styles. As instructors ourselves, we know that the balance we choose depends on the students we have: sometimes a focus on conceptual understanding is best; sometimes more skill-building is needed.

To enable instructors to select the balance appropriate for their students, the fifth edition expands the options available for customizing material. For example, we have integrated examples involving linear inequalities in our treatment of linear equations, and offered more examples on rates of change in several chapters. The sections involving limit notation and phase shift have been restructured so that these topics can be easily skipped. Transformations of functions are now introduced in Chapter 2, while the full treatment remains in Chapter 6, providing instructors with more freedom about when to introduce them and how deeply to cover this topic.

Origin of Text: The Calculus Consortium for Higher Education

This book is the work of faculty at a diverse consortium of institutions, and was originally generously supported by the National Science Foundation. It represents the first consensus among such a diverse group of faculty to have shaped a mainstream precalculus text. Bringing together the results of research and experience with the views of many users, this text is designed to be used in a wide range of institutions.

Guiding Principles: Varied Problems and the Rule of Four

Conceptual understanding is enhanced when students engage in non-procedural problem-solving. Problems classified into clear-cut types tend to develop proficiency at finding answers, but not necessarily a robust conceptual understanding. Strong problem-solvers feel capable of making progress on problems they have not seen before, not just those of known type. Consequently, we are guided by the following principles:

- Problems are varied and often challenging. Many of our problems cannot be answered by following a template in the text.
- Text examples are diverse and represent the natural integration of skills and concepts.
- The Rule of Four: each concept and function is represented symbolically, numerically, graphically, and verbally. This principle, originally introduced by the Consortium, promotes multiple representations.

- We provide students and instructors quick means of assessing comprehension before moving on by using the true-false Strengthen Your Understanding problems at the end of each chapter.
- Our central theme, functions as models of change, links the components of our precalculus curriculum. Algebra is integrated where appropriate.
- Topics are fewer in number than it is customary so they can be treated in greater depth. The topics are those essential to the study of calculus.
- Problems involving real data give students practice with modeling.
- We include both classic and current applications to prepare students to model with mathematics in a variety of contexts.
- Our problems allow students to become proficient in the use of technology, including symbolic manipulators, computers, tablets, and online software, as appropriate.
- Our precalculus materials allow for a broad range of teaching styles. They are flexible enough for use in large lecture halls, small classes, or in group or lab settings.

Changes in the Fifth Edition

The fifth edition reflects the many helpful suggestions from users while preserving the focus and guiding principles of previous editions. We have made the following changes:

- **Many examples and problems** are new or have been rewritten. Data has been updated and new data introduced.
- **The three chapters on trigonometry** have been reorganized and rewritten:
  - **Trigonometric functions on the circle** are introduced both in radians and degrees in Chapter 7, highlighting the natural relationship between radians and the unit circle as soon as periodic functions are introduced.
  - **Sinusoidal functions** modeling periodic phenomena and **simple trigonometric equations** are also introduced in Chapter 7, while more involved trigonometric models and equations requiring relationships such as **double-angle identities** are treated in Chapter 9.
  - **Phase shift** has been made an optional topic in Chapter 7.
  - **Triangle trigonometry** is discussed separately in Chapter 8, after trigonometric and sinusoidal functions are treated in Chapter 7.
- A brief exploration of **linear inequalities** has been integrated with the material on solving linear equations in Chapter 1.
- **Vertical and horizontal shifts** are introduced in Chapter 2, and referenced in Chapter 3 when introducing the vertex form of a quadratic equation. Shifts are reviewed at the beginning of Chapter 6 and considered in combination with other transformations.
- **Odd and even functions** are introduced by looking at invariance of certain functions under reflections, and thus better integrated with the transformation focus of Chapter 6.
- The effect of **changing the order of transformations**, the last section in Chapter 6 in the fourth edition, has been shortened and included in the previous section.
- The section on **power functions** in Chapter 11 has been rewritten to increase the focus on graphical behavior and proportionality.
- **Examples on average rate of change** have been added throughout the book, such as for exponential functions (in Chapter 5) and periodic functions (in Chapters 7 and 9).
- **WileyPLUS**, the primary online resource suite paired with the textbook, has been updated with improved problems and hints, including many new problems from the fifth edition.

What Student Background is Expected?

Students using this book should have successfully completed a course in intermediate algebra or high school algebra II. The book is thought-provoking for well-prepared students while still accessible to students with weaker backgrounds. Providing numerical, graphical, and algebraic approaches builds on different student strengths and provides students with a variety of ways to master the material. Multiple representations give students tools to persist, lowering failure rates.

Our Experiences

The first four editions of this book were used at hundreds of schools around the country in a wide variety of settings. It has been used successfully in both semester and quarter systems, in large lectures and small classes as well as in full-year courses in secondary schools. It has also been used in computer labs and small groups, often with the integration of a number of different technologies.

Content

The central theme of this book is functions as models of change. We emphasize that functions can be grouped into families and that functions can be used as models. We explore how function characteristics connect to difference quotients and rates of change, naturally previewing key calculus ideas.

Because linear, quadratic, exponential, power, and periodic functions are most frequently used to model physical phenomena, they are introduced before polynomial and rational functions. Once introduced, a family of functions is compared and contrasted with other families of functions.

A large number of the examples that students see in this precalculus course are real-world problems. By the end of the course, we hope that students will use functions to help them understand the world in which they live. We include non-routine problems to emphasize that such problems are not only part of mathematics, but in some sense are the reason for doing mathematics.

Technology

The book does not require any specific software or technology. Instructors have used the material with graphing calculators, graphing software, or scientific calculators.

Chapter 1: Linear Functions and Change

This chapter introduces the concept of a function as well as graphical, tabular, symbolic, and verbal representations of functions, discussing the advantages and disadvantages of each representation. It introduces rates of change and uses them to characterize linear functions. Examples on modeling with linear functions, including interpreting linear inequalities, are discussed. A section on fitting a linear function to data is included.

The **Skills Refresher** section for Chapter 1 reviews linear equations, linear inequalities, and the coordinate plane.

Chapter 2: Functions

This chapter studies functions in more detail. It introduces domain, range, previews function shifts and the concepts of composite and inverse functions, and investigates the idea of concavity using rates of change. A section on piecewise functions is included.

Chapter 3: Quadratic Functions

This chapter introduces the standard, factored, and vertex forms of a quadratic function and explores their relationship to graphs, including shifts. The family of quadratic functions provides an opportunity to see the effect of parameters on functional behavior.

The **Skills Refresher** section for Chapter 3 reviews factoring, completing the square, and quadratic equations.

Chapter 4: Exponential Functions

This chapter introduces the family of exponential functions and the number e. It compares exponential and linear functions, solves exponential equations graphically, and gives applications to compound interest.

The **Skills Refresher** section for Chapter 4 reviews the properties of exponents.

Chapter 5: Logarithmic Functions

This chapter introduces logarithmic functions with base 10 and base e, both in order to solve exponential equations and as inverses of exponential functions. After discussing manipulations with logarithms, the chapter focuses on modeling with exponential functions and logarithms. Logarithmic scales and a section on linearizing data conclude the chapter.

The **Skills Refresher** section for Chapter 5 reviews the properties of logarithms.

Chapter 6: Transformations of Functions and Their Graphs

This chapter investigates transformations. It revisits shifts, introduces reflections and stretches, and explores even and odd symmetry using transformations. The effect of ordering when combining transformations is explored.

Chapter 7: Trigonometry and Periodic Functions

This chapter introduces trigonometric functions in both radians and degrees as models for periodic motion. After exploring graphs and formulas of sine, cosine, and tangent, general sinusoidal functions are introduced. This chapter introduces the inverse trigonometric functions to solve trigonometric equations.

The **Skills Refresher** section for Chapter 7 reviews sine and cosine values of special angles measured in both radians and degrees.

Chapter 8: Triangle Trigonometry and Polar Coordinates

This chapter develops right-triangle trigonometry, and introduces the Law of Sines and the Law of Cosines. It also defines polar coordinates, explores their relationship to Cartesian coordinates, and considers the graphs of polar inequalities.

Note: Chapter 8 can be skipped by instructors who prefer not to emphasize triangle trigonometry.

Chapter 9: Trigonometric Identities, Models and Complex Numbers

This chapter opens with a discussion of trigonometric equations, and then explores trigonometric identities and their role in trigonometric models, such as damped oscillations and acoustic beats. The chapter closes with complex numbers, including Euler's Formula and de Moivre's theorem.

Chapter 10: Compositions, Inverses, and Combinations of Functions

This chapter discusses combinations of functions. It investigates composite and inverse functions, which were introduced in Chapter 2, in more detail.

Chapter 11: Polynomial and Rational Functions

This chapter discusses power functions, polynomials, and rational functions. The chapter explores dominance and long-run behavior and concludes by comparing polynomial and exponential functions, and by fitting functions to data.

The **Skills Refresher** section for Chapter 11 reviews algebraic fractions.

Chapter 12: Vectors

This chapter contains material on vectors and operations involving vectors. An introduction to matrices is included in the last section.

Chapter 13: Sequences and Series

This chapter introduces arithmetic and geometric sequences and series and their applications.

Chapter 14: Parametric Equations and Conic Sections

The concluding chapter looks at parametric equations, implicit functions, hyperbolic functions, and conic sections: circles, ellipses, and hyperbolas. The chapter includes a section on the geometrical properties of the conic sections and their applications to orbits.

Note: Chapter 14 is available online only.

Supplementary Materials

The following supplementary materials are available for the fifth edition:

- **The Instructor's Manual** contains teaching tips, lesson plans, syllabi, and worksheets. It has been expanded and revised to include worksheets, identification of technology-oriented problems, and new syllabi. (ISBN 978-1-119-01383-9)
- **The Printed Test Bank** contains test questions arranged by section.
- **The Instructor's Solution Manual** has complete solutions to all problems. (ISBN 978-1-118-94162-1)
- **The Student Solution Manual** has complete solutions to half of the odd-numbered problems. (ISBN 978-1-118-94163-8)
- **The Computerized Test Bank**, available in both PC and Macintosh formats, allows instructors to create, customize, and print a test containing any combinations of questions from a large bank of questions. Instructors can also customize the questions or create their own.
- **Classroom Activities** are posted at the book companion website. These activities were developed to facilitate in-class group work as well as to introduce new concepts and to practice skills. In addition to the blank copies for each activity that can be handed out to the students, a copy of the activity with fully worked out solutions is also available.
- **The Book Companion Site** at www.wiley.com/college/connally contains all instructor supplements.
- **WileyPLUS** is a powerful online suite of teaching and learning resources tightly integrated with the text. WileyPLUS enables instructors to assign, deliver and grade individually customized homework assignments using exercises and problems from the text. Students receive immediate feedback on their homework and access to full solutions to assigned problems electronically. Students may also access hints to the problems. In addition to online homework, WileyPLUS provides student tutorials, an instructor gradebook, integrated links to the electronic version of the text, and all of the text's supplemental materials. For more information, visit www.wiley.com/college/wileyplus or contact your local Wiley representative for more details.
- **Mini-lecture Videos** linked with examples in the WileyPLUS student version of the text provide greater detail to the solution of examples in each section of the text. These may assist students in reading the text prior to class or in reviewing material after class.
- **The Faculty Network** is a peer-to-peer network of academic faculty dedicated to the effective use of technology in the classroom. This group can help you apply innovative classroom techniques, implement specific software packages, and tailor the technology experience to the specific needs of each individual class. Visit www.wherefacultyconnect.com or ask your Wiley representative for details.

ConcepTests

ConcepTests, modeled on the pioneering work of Harvard physicist Eric Mazur, are questions designed to promote active learning during class, particularly (but not exclusively) in large lectures. Our evaluation data show students taught with ConcepTests outperformed students taught by traditional lecture methods 73% versus 17% on conceptual questions, and 63% versus 54% on computational problems. ConcepTests arranged by section are available in PowerPoint and Classroom Response System-ready formats from your Wiley representative. (ISBN 978-1-118-94161-4)

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Jerry Morris	Karen Rhea	Ellen Schmierer	Pat Shure
Adam H. Spiegler	Carl Swenson	Aaron D. Wootton	

To Students: How to Learn from this Book

- This book may be different from other math textbooks that you have used, so it may be helpful to know about some of the differences in advance. At every stage, this book emphasizes the *meaning* (in practical, graphical or numerical terms) of the symbols you are using. There is much less emphasis on "plug-and-chug" and using formulas, and much more emphasis on the interpretation of these formulas than you may expect. You will often be asked to explain your ideas in words or to explain an answer using graphs.
- The book contains the main ideas of precalculus in plain English. Success in using this book will depend on reading, questioning, and thinking hard about the ideas presented. It will be helpful to read the text in detail, not just the worked examples.
- There are few examples in the text that are exactly like the homework problems, so homework problems can't be done by searching for similar-looking "worked out" examples. Success with the homework will come by grappling with the ideas of precalculus.
- Many of the problems in the book are open-ended. This means that there is more than one correct approach and more than one correct solution. Sometimes, solving a problem relies on common-sense ideas that are not stated in the problem explicitly but which you know from everyday life.
- This book assumes that you have access to a calculator or computer that can graph functions and find (approximate) roots of equations. There are many situations where you may not be able to find an exact solution to a problem, but can use a calculator or computer to get a reasonable approximation. An answer obtained this way can be as useful as an exact one. However, the problem does not always state that a calculator is required, so use your own judgment.
- This book attempts to give equal weight to four methods for describing functions: graphical (a picture), numerical (a table of values), algebraic (a formula) and verbal (words). Sometimes it's easier to translate a problem given in one form into another. For example, you might replace the graph of a parabola with its equation, or plot a table of values to see its behavior. It is important to be flexible about your approach: if one way of looking at a problem doesn't work, try another.
- Students using this book have found discussing these problems in small groups helpful. There are a great many problems that are not cut-and-dried; it can help to attack them with the other perspectives your colleagues can provide. If group work is not feasible, see if your instructor can organize a discussion session in which additional problems can be worked on.
- You are probably wondering what you'll get from the book. The answer is, if you put in a solid effort, you will get a real understanding of functions as well as a real sense of how mathematics is used in the age of technology.

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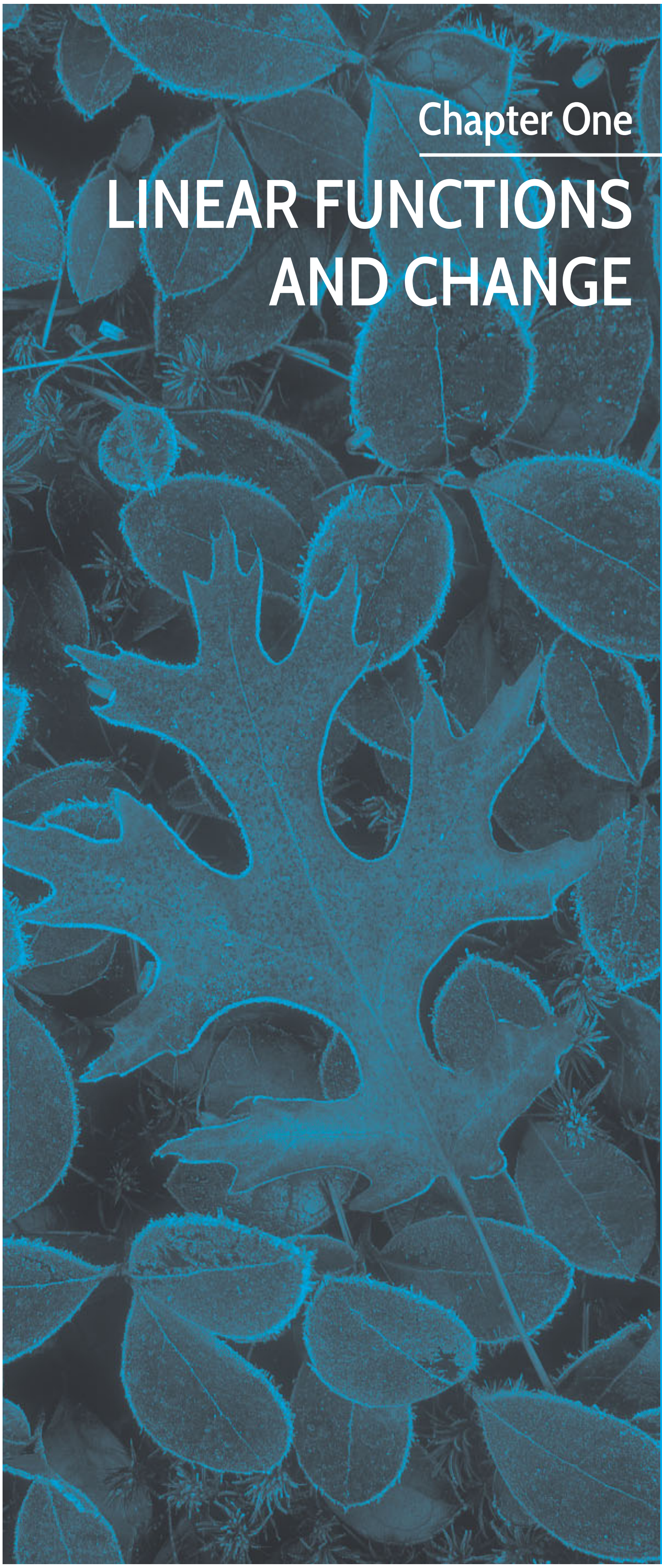


Chapter One

LINEAR FUNCTIONS AND CHANGE

Contents

1.1 FUNCTIONS AND FUNCTION NOTATION

In everyday language, the word *function* expresses the notion of dependence. For example, a person might say that election results are a function of the economy, meaning that the winner of an election is determined by how the economy is doing. Someone else might claim that car sales are a function of the weather, meaning that the number of cars sold on a given day is affected by the weather.

In mathematics, the meaning of the word *function* is more precise, but the basic idea is the same. A function is a relationship between two quantities. If the value of the first quantity determines exactly one value of the second quantity, we say the second quantity is a function of the first. We make the following definition:

> A **function** is a rule that takes certain numbers as inputs and assigns to each input number exactly one output number. The output is a function of the input.

The inputs and outputs are also called *variables*.

Representing Functions: Words, Tables, Graphs, and Formulas

A function can be described using words, data in a table, points on a graph, or a formula.

Example 1 It is a surprising biological fact that most crickets chirp at a rate that increases as the temperature increases. For the snowy tree cricket (*Oecanthus fultoni*), the relationship between temperature and chirp rate is so reliable that this type of cricket is called the thermometer cricket. We can estimate the temperature (in degrees Fahrenheit) by counting the number of times a snowy tree cricket chirps in 15 seconds and adding 40. For instance, if we count 20 chirps in 15 seconds, then a good estimate of the temperature is $20 + 40 = 60°$F.[1]

The rule used to find the temperature T (in °F) from the chirp rate R (in chirps per minute) is an example of a function. The input is chirp rate and the output is temperature. Describe this function using words, a table, a graph, and a formula.

Solution

- **Words**: To estimate the temperature, we count the number of chirps in fifteen seconds and add forty. Alternatively, we can count R chirps per minute, divide R by four and add forty. This is because there are one-fourth as many chirps in fifteen seconds as there are in sixty seconds. For instance, 80 chirps per minute works out to $\frac{1}{4} \cdot 80 = 20$ chirps every 15 seconds, giving an estimated temperature of $20 + 40 = 60°$F.
- **Table**: Table 1.1 gives the estimated temperature, T, as a function of R, the number of chirps per minute. Notice the pattern in Table 1.1: each time the chirp rate, R, goes up by 20 chirps per minute, the temperature, T, goes up by 5°F.

Table 1.1 *Chirp rate and temperature*

R, chirp rate (chirps/minute)	T, predicted temperature (°F)
20	45
40	50
60	55
80	60
100	65
120	70
140	75
160	80

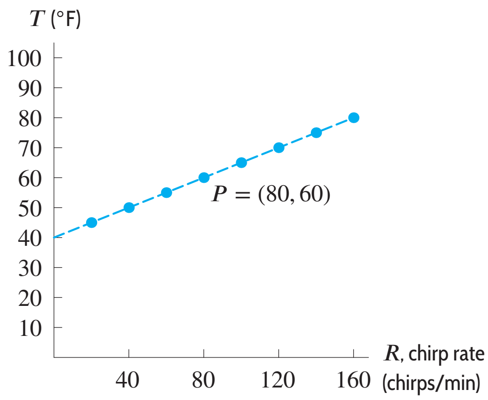


Figure 1.1: Chirp rate and temperature

[1] This relationship is often called Dolbear's Law, as it was first proposed in Amos Dolbear, "The Cricket as a Thermometer," in *The American Naturalist,* 31(1897), pp. 970–971.

- **Graph**: The data from Table 1.1 are plotted in Figure 1.1. For instance, the pair of values $R = 80$, $T = 60$ is plotted as the point P, which is 80 units along the horizontal axis and 60 units up the vertical axis. Data represented in this way are said to be plotted on the *Cartesian plane*. The precise position of P is shown by its coordinates, written $P = (80, 60)$.
- **Formula**: A formula is an equation giving T in terms of R. Dividing the chirp rate by four and adding forty gives the estimated temperature, so:

$$\underbrace{\text{Estimated temperature (in °F)}}_{T} = \frac{1}{4} \cdot \underbrace{\text{Chirp rate (in chirps/min)}}_{R} + 40.$$

Rewriting this using the variables T and R gives the formula:

$$T = \frac{1}{4}R + 40.$$

Let's check the formula. Substituting $R = 80$, we have

$$T = \frac{1}{4} \cdot 80 + 40 = 60,$$

which agrees with point $P = (80, 60)$ in Figure 1.1. The formula $T = \frac{1}{4}R + 40$ also tells us that if $R = 0$, then $T = 40$. Thus, the dashed line in Figure 1.1 crosses (or intersects) the T-axis at $T = 40$; we say the T*-intercept* is 40.

All the descriptions given in Example 1 provide the same information, but each description has a different emphasis. A relationship between variables is often given in words, as at the beginning of Example 1. Table 1.1 is useful because it shows the predicted temperature for various chirp rates. Figure 1.1 is more suggestive of a trend than the table, although it is harder to read exact values of the function. For example, you might have noticed that every point in Figure 1.1 falls on a straight line that slopes up from left to right. In general, a graph can reveal a pattern that might otherwise go unnoticed. Finally, the formula has the advantage of being both compact and precise. However, this compactness can also be a disadvantage since it may be harder to gain as much insight from a formula as from a table or a graph.

Mathematical Models

When we use a function to describe an actual situation, the function is referred to as a **mathematical model**. The formula $T = \frac{1}{4}R + 40$ is a mathematical model of the relationship between the temperature and the cricket's chirp rate. Such models can be powerful tools for understanding phenomena and making predictions. For example, this model predicts that when the chirp rate is 80 chirps per minute, the temperature is 60°F. In addition, since $T = 40$ when $R = 0$, the model predicts that the chirp rate is 0 at 40°F. Whether the model's predictions are accurate for chirp rates down to 0 and temperatures as low as 40°F is a question that mathematics alone cannot answer; an understanding of the biology of crickets is needed. However, we can safely say that the model does not apply for temperatures below 40°F, because the chirp rate would then be negative. For the range of chirp rates and temperatures in Table 1.1, the model is remarkably accurate.

In everyday language, saying that T is a function of R suggests that making the cricket chirp faster would somehow make the temperature change. Clearly, the cricket's chirping does not cause the temperature to be what it is. In mathematics, saying that the temperature "depends" on the chirp rate means only that knowing the chirp rate is sufficient to tell us the temperature.

Function Notation

To indicate that a quantity Q is a function of a quantity t, we abbreviate

$$Q \text{ is a function of } t \quad \text{to} \quad Q \text{ equals "} f \text{ of } t\text{"}$$

and, using function notation, to

$$Q = f(t).$$

Thus, applying the rule f to the input value, t, gives the output value, $f(t)$, which is a value of Q. Here Q is called the *dependent variable* and t is called the *independent variable*. In other words,

$$\text{Output} = f(\text{Input})$$

or

$$\text{Dependent} = f(\text{Independent}).$$

We could have used any letter, not just f, to represent the rule.

The expressions "Q depends on t" or "Q is a function of t" do *not* imply a cause-and-effect relationship, as the snowy tree cricket example illustrates.

Example 2 Example 1 gives the following formula for estimating air temperature, T, based on the chirp rate, R, of the snowy tree cricket:

$$T = \frac{1}{4}R + 40.$$

In this formula, T depends on R. Writing $T = f(R)$ indicates that the relationship is a function.

Example 3 The number of gallons of paint needed to paint a house depends on the size of the house. A gallon of paint typically covers 250 square feet. Thus, the number of gallons of paint, n, is a function of the area to be painted, A ft^2. We write $n = f(A)$.

(a) Find a formula for f.
(b) Explain in words what the statement $f(10{,}000) = 40$ tells us about painting houses.

Solution

(a) If $A = 250$, the house requires one gallon of paint. If $A = 500$, it requires $500/250 = 2$ gallons of paint, if $A = 750$ it requires $750/250 = 3$ gallons of paint, and so on. We see that a house of area A requires $A/250$ gallons of paint, so n and A are related by the formula

$$n = f(A) = \frac{A}{250}.$$

(b) The input of the function $n = f(A)$ is an area and the output is an amount of paint. The statement $f(10{,}000) = 40$ tells us that an area of $A = 10{,}000$ ft^2 requires $n = 40$ gallons of paint.

Functions Don't Have to Be Defined by Formulas

People sometimes think that functions are always represented by formulas. However, other representations, such as tables or graphs, can be useful.

Example 4 The average monthly rainfall, R, at Chicago's O'Hare airport is given in Table 1.2, where time, t, is in months and $t = 1$ is January, $t = 2$ is February, and so on. The rainfall is a function of the month, so we write $R = f(t)$. However, there is no formula that gives R when t is known. Evaluate $f(1)$ and $f(11)$. Explain what your answers mean.

Table 1.2 *Average monthly rainfall at Chicago's O'Hare airport*

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Rainfall, R (inches)	1.8	1.8	2.7	3.1	3.5	3.7	3.5	3.4	3.2	2.5	2.4	2.1

Solution The value of $f(1)$ is the average rainfall in inches at Chicago's O'Hare airport in a typical January. From the table, $f(1) = 1.8$ inches. Similarly, $f(11) = 2.4$ means that in a typical November, there are 2.4 inches of rain at O'Hare.

When Is a Relationship Not a Function?

It is possible for two quantities to be related and yet for neither quantity to be a function of the other.

Example 5 A national park contains foxes that prey on rabbits. Table 1.3 gives the two populations, F and R, over a 12-month period, where $t = 0$ means January 1, $t = 1$ means February 1, and so on.

Table 1.3 *Number of foxes and rabbits in a national park, by month*

t, month	0	1	2	3	4	5	6	7	8	9	10	11
R, rabbits	1000	750	567	500	567	750	1000	1250	1433	1500	1433	1250
F, foxes	150	143	125	100	75	57	50	57	75	100	125	143

(a) Is F a function of t? Is R a function of t?
(b) Is F a function of R? Is R a function of F?

Solution

(a) Remember that for a relationship to be a function, an input can only give a single output. Both F and R are functions of t. For each value of t, there is exactly one value of F and exactly one value of R. For example, Table 1.3 shows that if $t = 5$, then $R = 750$ and $F = 57$. This means that on June 1 there are 750 rabbits and 57 foxes in the park. If we write $R = f(t)$ and $F = g(t)$, then $f(5) = 750$ and $g(5) = 57$.

(b) No, F is not a function of R. For example, suppose $R = 750$, meaning there are 750 rabbits. This happens both at $t = 1$ (February 1) and at $t = 5$ (June 1). In the first instance, there are 143 foxes; in the second instance, there are 57 foxes. Since there are R-values which correspond to more than one F-value, F is not a function of R.

Similarly, R is not a function of F. At time $t = 5$, we have $R = 750$ when $F = 57$, while at time $t = 7$, we have $R = 1250$ when $F = 57$ again. Thus, the value of F does not uniquely determine the value of R.

How to Tell if a Graph Represents a Function: Vertical Line Test

What does it mean graphically for y to be a function of x? Look at a graph of y against x, with y on the vertical axis and x on the horizontal axis. For a function, each x-value corresponds to exactly one y-value. This means that the graph intersects any vertical line at most once (either once or not at all). If a vertical line cuts the graph twice, the graph contains two points with different y-values but the same x-value; this violates the definition of a function. Thus, we have the following criterion:

Vertical Line Test: If there is a vertical line that intersects a graph in more than one point, then the graph does not represent a function.

Example 6 In which of the graphs in Figures 1.2 and 1.3 could y be a function of x?

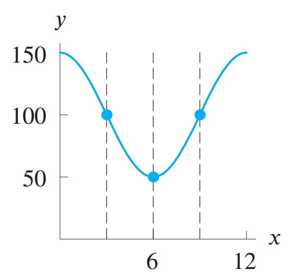


Figure 1.2: Since no vertical line intersects this curve at more than one point, y could be a function of x

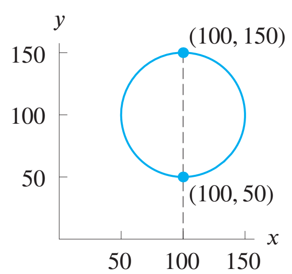


Figure 1.3: Since one vertical line intersects this curve at more than one point, y is not a function of x

Solution The graph in Figure 1.2 could represent y as a function of x because no vertical line intersects this curve in more than one point. The graph in Figure 1.3 does not represent a function because the vertical line shown intersects the curve at two points.

A graph fails the vertical line test if at least one vertical line cuts the graph more than once, as in Figure 1.3. However, if a graph represents a function, then *every* vertical line must intersect the graph at no more than one point.

Exercises and Problems for Section 1.1

Skill Refresher

In Exercises S1–S4, simplify each expression.

S1. $c + \frac{1}{2}c$

S2. $P + 0.07P + 0.02P$

S3. $2\pi r^2 + 2\pi r \cdot 2r$

S4. $\frac{12\pi - 2\pi}{6\pi}$

In Exercises S5–S8, find the value of the expressions for the given value of x and y.

S5. $x - 5y$ for $x = \frac{1}{2}$, $y = -5$.

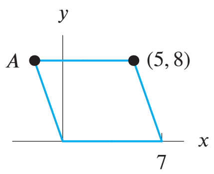
S6. $1 - 12x + x^2$ for $x = 3$.

S7. $\frac{3}{2 - x^3}$ for $x = -1$.

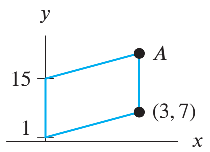
S8. $\frac{4}{1 + 1/x}$ for $x = -\frac{3}{4}$.

The figures in Exercises S9–S10 are parallelograms. Find the coordinates of the point A.

S9.



S10.



Exercises

In Exercises 1–2, write the relationship using function notation (that is, y is a function of x is written $y = f(x)$).

1. Number of molecules, m, in a gas, is a function of the volume of the gas, v.

2. Weight, w, is a function of caloric intake, c.

In Exercises 3–6, label the axes for a sketch to illustrate the given statement.

3. "Over the past century we have seen changes in the population, P (in millions), of the city. . ."

4. "Sketch a graph of the cost of manufacturing q items. . ."

5. "Graph the pressure, p, of a gas as a function of its volume, v, where p is in pounds per square inch and v is in cubic inches."

6. "Graph D in terms of y. . ."

7. Figure 1.4 gives the depth of the water at Montauk Point, New York, for a day in November.
   (a) How many high tides took place on this day?
   (b) How many low tides took place on this day?
   (c) How much time elapsed in between high tides?

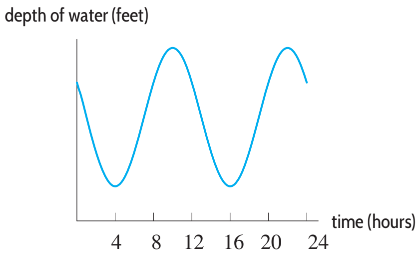


Figure 1.4

8. Using Table 1.4, graph $n = f(A)$, the number of gallons of paint needed to cover walls of area A. Identify the independent and dependent variables.

Table 1.4

A	0	250	500	750	1000	1250	1500
n	0	1	2	3	4	5	6

9. Use Figure 1.5 to fill in the missing values:

(a) $f(0) = ?$ **(b)** $f(?) = 0$

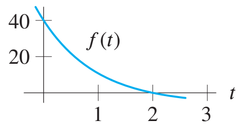


Figure 1.5

10. Use Table 1.5 to fill in the missing values. (There may be more than one answer.)

(a) $f(0) = ?$ **(b)** $f(?) = 0$

(c) $f(1) = ?$ **(d)** $f(?) = 1$

Table 1.5

x	0	1	2	3	4
$f(x)$	4	2	1	0	1

Exercises 11–14 use Figure 1.6.

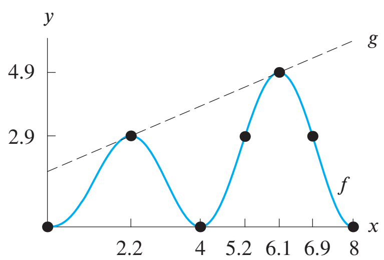


Figure 1.6

11. Find $f(6.9)$.

12. Give the coordinates of two points on the graph of g.

13. Solve $f(x) = 0$ for x.

14. Solve $f(x) = g(x)$ for x.

15. **(a)** You are going to graph $p = f(w)$. Which variable goes on the horizontal axis?

(b) If $10 = f(-4)$, give the coordinates of a point on the graph of f.

(c) If 6 is a solution of the equation $f(w) = 1$, give a point on the graph of f.

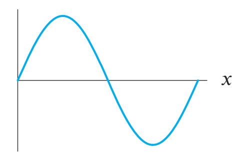
In Exercises 16–19 a relationship is given between two quantities. Are both quantities functions of the other one, or is one or neither a function of the other? Explain.

16. $7w^2 + 5 = z^2$ **17.** $y = x^4 - 1$ **18.** $m = \sqrt{t}$

19. The number of gallons of gas, g, at \$3 per gallon and the number of pounds of coffee, c, at \$10 per pound that can be bought for a total of \$100.

In Exercises 20–25, could the graph represent y as a function of x?

20.



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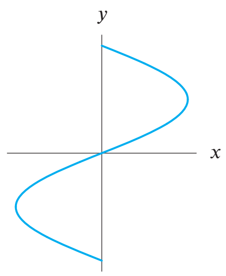

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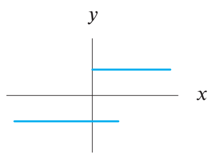

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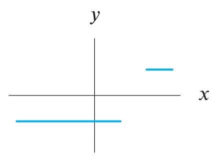

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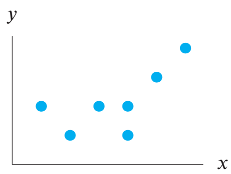

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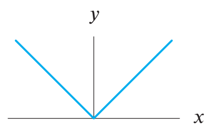


Problems

26. At the end of a semester, students' math grades are listed in a table which gives each student's ID number in the left column and the student's grade in the right column. Let N represent the ID number and G represent the grade. Which quantity, N or G, must necessarily be a function of the other?

27. A person's blood sugar level at a particular time of the day is partially determined by the time of the most recent meal. After a meal, blood sugar level increases rapidly, then slowly comes back down to a normal level. Sketch a person's blood sugar level as a function of time over the course of a day. Label the axes to indicate normal blood sugar level and the time of each meal.

28. When a parachutist jumps out of a plane, the speed of her fall increases until she opens her parachute, at which time her falling speed suddenly decreases and stays constant until she reaches the ground. Sketch a possible graph of the height H of the parachutist as a function of time t, from the time when she jumps from the plane to the time when she reaches the ground.

29. A buzzard is circling high overhead when it spies some road kill. It swoops down, lands, and eats. Later it takes off sluggishly, and resumes circling overhead, but at a lower altitude. Sketch a possible graph of the height of the buzzard as a function of time.

30. Table 1.6 gives the ranking r for three different names—Hannah, Alexis, and Madison. Of the three names, which was most popular and which was least popular in

(a) 1995? **(b)** 2004?

Table 1.6 *Ranking of names—Hannah (r_h), Alexis (r_a), and Madison (r_m)—for girls born between 1995 ($t = 0$) and 2004 ($t = 9$)*[2]

t	0	1	2	3	4	5	6	7	8	9
r_h	7	7	5	2	2	2	3	3	4	5
r_a	14	8	8	6	3	6	5	5	7	11
r_m	29	15	10	9	7	3	2	2	3	3

31. Table 1.6 gives information about the popularity of the names Hannah, Madison, and Alexis. Describe in words what your answers to parts (a)–(c) tell you about these names.

(a) Evaluate $r_m(0) - r_h(0)$.
(b) Evaluate $r_m(9) - r_h(9)$.
(c) Solve $r_m(t) < r_a(t)$.

32. Figure 1.7 shows the fuel consumption (in miles per gallon, mpg) of a car traveling at various speeds (in mph).

(a) How much gas is used on a 300-mile trip at 40 mph?
(b) How much gas is saved by traveling 60 mph instead of 70 mph on a 200-mile trip?
(c) According to this graph, what is the most fuel-efficient speed to travel? Explain.

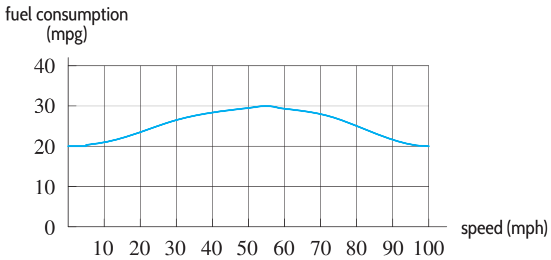
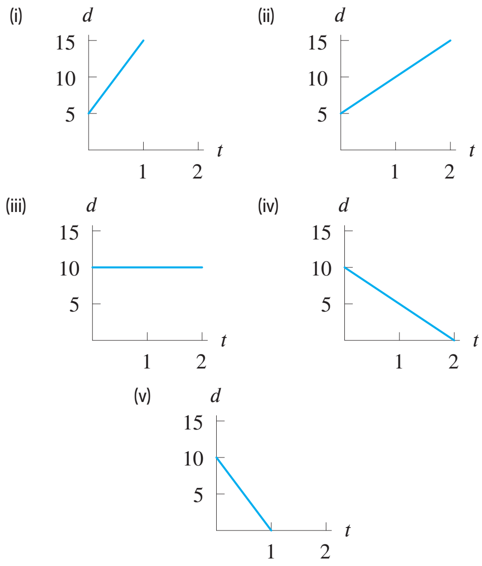


Figure 1.7

33. Match each story about a bike ride to one of the graphs (i)–(v), where d represents distance from home and t is time in hours since the start of the ride. (A graph may be used more than once.)

(a) Starts 5 miles from home and rides 5 miles per hour away from home.
(b) Starts 5 miles from home and rides 10 miles per hour away from home.
(c) Starts 10 miles from home and arrives home one hour later.
(d) Starts 10 miles from home and is halfway home after one hour.
(e) Starts 5 miles from home and is 10 miles from home after one hour.



34. Figure 1.8 shows the mass of water in air, in grams of water per kilogram of air, as a function of air temperature in °C, for two different levels of relative humidity.

(a) Find the mass of water in 1 kg of air at 30°C if the relative humidity is

(a) 100% **(b)** 50% **(c)** 75%

(b) How much water is in a room containing 300 kg of air if the relative humidity is 50% and the temperature is 20°C?
(c) The density of air is approximately 1.2 kg/m^3. If the relative humidity in your classroom is 50% and the temperature is 20°C, estimate the amount of water in the air.

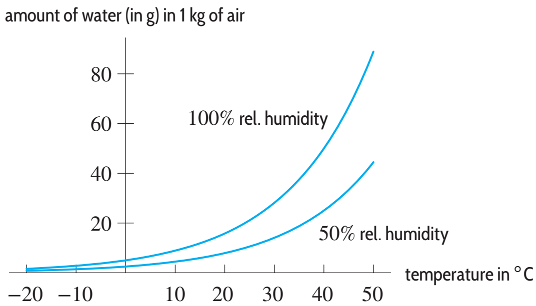


Figure 1.8

[2]Data from the SSA website at www.ssa.gov, accessed January 12, 2006.

35. Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements.

(a) $f(10) = 100.3$ **(b)** $f(a) = 20$
(c) $f(20) = b$ **(d)** $n = f(t)$

36. **(a)** Ten inches of snow is equivalent to about one inch of rain.[3] Write an equation for the amount of precipitation, measured in inches of rain, $r = f(s)$, as a function of the number of inches of snow, s.
(b) Evaluate and interpret $f(5)$.
(c) Find s such that $f(s) = 5$ and interpret your result.

37. An 8-foot-tall cylindrical water tank has a base of diameter 6 feet.

(a) How much water can the tank hold?
(b) How much water is in the tank if the water is 5 feet deep?
(c) Write a formula for the volume of water as a function of its depth in the tank.

38. Table 1.7 gives $A = f(d)$, the amount of money in bills of denomination d circulating in US currency in 2013.[4] For example, there were \$74.5 billion worth of \$50 bills in circulation.

(a) Find $f(100)$. What does this tell you about money?
(b) Are there more \$1 bills or \$5 bills in circulation?

Table 1.7

Denomination (\$)	1	2	5	10	20	50	100
Circulation (\$bn)	10.6	2.1	12.7	18.5	155.0	74.5	924.7

39. Table 1.8 shows the daily low temperature for a one-week period in New York City during July.

(a) What was the low temperature on July 19?
(b) When was the low temperature 73°F?
(c) Is the daily low temperature a function of the date?
(d) Is the date a function of the daily low temperature?

Table 1.8

Date	17	18	19	20	21	22	23
Low temp (°F)	73	77	69	73	75	75	70

40. Use the data from Table 1.3 on page 5.

(a) Plot R on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that R is a function of t.
(b) Plot F on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that F is a function of t.
(c) Plot F on the vertical axis and R on the horizontal axis. From this graph show that F is not a function of R.
(d) Plot R on the vertical axis and F on the horizontal axis. From this graph show that R is not a function of F.

41. Since Roger Bannister broke the 4-minute mile on May 6, 1954, the record has been lowered by over sixteen seconds. Table 1.9 shows the year and times (as min:sec) of new world records for the one-mile run.[5] (Official records for the mile ended in 1999.)

(a) Is the time a function of the year? Explain.
(b) Is the year a function of the time? Explain.
(c) Let $y(r)$ be the year in which the world record, r, was set. Explain what is meant by the statement $y(3:47.33) = 1981$.
(d) Evaluate and interpret $y(3:51.1)$.

Table 1.9

Year	Time	Year	Time	Year	Time
1954	3:59.4	1966	3:51.3	1981	3:48.53
1954	3:58.0	1967	3:51.1	1981	3:48.40
1957	3:57.2	1975	3:51.0	1981	3:47.33
1958	3:54.5	1975	3:49.4	1985	3:46.32
1962	3:54.4	1979	3:49.0	1993	3:44.39
1964	3:54.1	1980	3:48.8	1999	3:43.13
1965	3:53.6				

42. The sales tax on an item is 6%. Express the total cost, C, in terms of the price of the item, P.

43. A price increases 5% due to inflation and is then reduced 10% for a sale. Express the final price as a function of the original price, P.

44. Write a formula for the area of a circle as a function of its radius and determine the percent increase in the area if the radius is increased by 10%.

45. There are x male job-applicants at a certain company and y female applicants. Suppose that 15% of the men are accepted and 18% of the women are accepted. Write an expression in terms of x and y representing each of the following quantities:

(a) The total number of applicants to the company.
(b) The total number of applicants accepted.
(c) The percentage of all applicants accepted.

[3] http://mo.water.usgs.gov/outreach/rain, accessed May 7, 2006.
[4] http://www.federalreserve.gov/paymentsystems/coin_currcircvalue.htm, accessed February 16, 2014.
[5] www.infoplease.com/ipsa/A0112924.html, accessed January 15, 2006.